Indian J. Phys. **78B(1)**, 31-38 (2004)

AN INTERNATIONAL JOURNAL IJP B

Energy and amplification of whistler mode wave

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Recieved 2 June 2003, accepted 12 November 2003

Abstract : Parallel resonance energy and wave growth/damping of the electrons in the magnetosphere at various frequencies for different *L*-values have been computed for whistler mode wave in the presence of thermal velocity and parallel electric field. The resonance energy of the electron decreases with wave frequency as well as with *L*-value. The wave growth/damping depends on the nature of the dc electric field present in the Earth's magnetosphere.

Keywords : Parallel resonance energy, wave-particle interaction, whistler-mode wave, wave growth/damping.

PACS Nos. : 94.20.Rr, 94.30.Hn, 94.30.Tz

1. Introduction

Wave-particle interactions play crucial role in the formation of the magnetopause boundary layer, generation of chorus and plasmaspheric hiss emission, precipitation of particles causing auroras *etc* [1]. Often the magnetospheric plasma is in a turbulent state. Electromagnetic waves are continuously observed mainly inside the plasmasphere and they cause continuous precipitation (drizzle) of particles. The trapped particle population almost always exhibits a marked anisotropy favourable in particular to cyclotron instability. New particles are continuously injected at different latitudes and radial distance *via* slow processes such as radial convection or diffusion and azimuthally drifts, which are the long-term consequences of more violent injections occurring during substroms. Therefore, a real geophysical situation may exist at a given place which lack dynamic equilibrium whence particles/waves are continuously injected/generated and precipitated/absorbed.

Low frequency waves interacting with charged particles in the magnetosphere can transport energy from one region to another [1]. The resonant interaction between energetic particles and whistler mode wave is responsible for the ELF/VLF emissions. This has been studied to explain the characteristic features of ELF/VLF emissions. The non-linear effects have also been included to explain the fine structures of frequency-time diagram of ELF/VLF emissions [2]. Brice [3] has shown that for gyrofrequency interaction between whistler modes and energetic electrons, the fractional decrease in transverse energy of the electrons is greater than the fractional decrease in their total energy. From energy considerations, Brice [3] has shown that the interaction is most probable in the equatorial region where parallel energy on a particular field line is minimum. Further, Inan [4] has shown that the waves grow in intensity with decrease in pitch angle of the electron causing precipitation of energetic electrons. The resonance velocity of interacting electrons and growth rate of the wave depends upon pitch angle [5,6]. Kennel and Petschek [7] first pointed out that for electron cyclotron plasma instability to take place it is necessary that the

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electron pitch angle distribution must be sufficiently anisotropic, *i.e.* there must be more energy in the gyroresonant electrons motion transverse to the geomagnetic field than along it. Thus, the anisotropy A_T must exceed a critical value.

Studies of Earth magnetosphere have indicated the presence of large scale direct current (dc) electric field and many theoretical models have been invoked to explain their origin and structure [8]. These fields have components both parallel and transverse to the geomagnetic field lines and at times may show oscillatory behaviour. The amplitudes of these electric fields range from a few mV/m to several hundred mV/m. These electric fields play a key role in the transfer of energy and particles from the solar wind to the magnetosphere and in the transport and acceleration of charge particles in the magnetosphere. The electrostatic field affects emission and propagation of whistler waves [9]. The effect of parallel electric field on wave propagation has been included through the modification of thermal temperature in zero order distribution function. The effect of parallel and transverse field on whistler wave propagation through an isothermal magnetoplasma has been discussed by Hsieh [10]. Misra *et al* [11] and Das and Singh [9] included the effect of plasma anisotropy in presence of parallel electric field on whistler wave propagation.

2. Dispersion relation

It is well known that the dispersion relation of the wave is derived from the coupled equation of motion for the charged particle and Maxwell's equations for the wave field. These equations are as follows :

$$\frac{\partial f}{\partial t}+\boldsymbol{v}.\frac{\partial f}{\partial \boldsymbol{v}}-\frac{e}{m}(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}).\frac{\partial f}{\partial \boldsymbol{v}}=\upsilon\left(f-f_0\right), \tag{1}$$

$$\nabla\times\boldsymbol{E}=-\frac{\partial\boldsymbol{B}}{\partial t}, \tag{2}$$

$$\nabla\times\boldsymbol{H}=\varepsilon_0\frac{\partial\boldsymbol{E}}{\partial t}-e\int \boldsymbol{v}fd^3v, \tag{3}$$

$$\nabla\boldsymbol{E}=-\frac{e}{\varepsilon_0}\int fd^3v, \tag{4}$$

where e and m are electronic charge and mass, υ is the electron collision frequency. Das and Singh [9] solved these equations by perturbation technique and derived the dispersion relation for whistler mode wave in presence of dc electric field (parallel to the ambient geomagnetic field) as

$$\frac{c^2k^2}{\omega_P^2}+\frac{k^2v_{t_\parallel}^2}{(\omega-\omega_H)^2}\left(A_T+\frac{\omega}{\omega-\omega_H}\right)+\frac{\omega}{\omega-\omega_H}$$
$$\left[\begin{array}{l}\left(\frac{\pi}{2}\right)^2\frac{\omega}{kv_{t_\parallel}}\left(A_T-\frac{\omega/\omega_H}{1-\omega/\omega_H}\right)\left(1-\frac{\omega}{\omega_H}\right)\exp\left(\xi^{-2}\right)+\\ +i\,\frac{eEk/\omega_{H}^2}{m\left(1-\omega/\omega_H\right)^2}\left(A_T+\frac{2\omega/\omega_H}{1-\omega/\omega_H}\right)-\left(\frac{\pi}{2}\right)^{1/2}\\ \frac{\omega_H}{kv_{t_\parallel}}\frac{eE}{mkv_{t_\parallel}^2}\left(2A_T-\frac{\omega/\omega_H}{1-\omega/\omega_H}\right)\left(1-\omega/\omega_H\right)\exp\left(-\xi^2\right)\end{array}\right]=0 \tag{5}$$

where

c = velocity of light in free space, E = parallel electric field, k = wave vector, ω = frequency of the wave, ω_P = electron plasma frequency, ω_H = electron grofrequency, $v_{t\parallel} = (k_BT_\parallel/m)^{1/2}$ is the thermal velocity of the electron, $A_T = (\frac{T_\perp}{T_\parallel}-1)$ is the temperature anisotropy, $\xi=\frac{(\omega-\omega_H)\alpha_\parallel^{1/2}}{\tilde{k}}$ is the argument of plasma dispersion function, $\tilde{k}=k-i\frac{eE}{k_BT}$ and $\alpha_\parallel = m/2k_BT_\parallel$.

3. Results and discussion

The interaction between energetic electrons and whistler mode waves (moving in opposite direction along geomagnetic field lines) are presented which are believed to be useful in the interpretation of observed geophysical phenomena such as discrete ELF and VLF emissions, emissions triggered by whistler mode waves launched by VLF transmitters, precipitating energetic electrons from the Van Allen radiation belts, and their atmospheric manifestations.

(i) Resonance energy :

The gyroresonance condition occurs when the Doppler-shifted wave frequency becomes equal to the electron gyrofrequency, which is expressed as

$$\omega-\boldsymbol{k}.\boldsymbol{v}=n\frac{\omega_H}{\beta}, \tag{6}$$

where, $\beta = (1 - v_\parallel^2/c^2\cos^2\alpha)^{-1/2}$, v is the electron velocity (with $v_\parallel$ as the parallel component of the electron's velocity along the geomagnetic field lines) and α the particle pitch angle at the equator, and $n = 0, \pm 1, \pm 2, \ldots\ldots\ldots$ is an integer. The case of $n = 0$ corresponds to the Landau resonance. For propagation along the geomagnetic field lines or ducted mode of propagation, the equation (6) becomes

$$\omega - kv_\parallel = \frac{n\omega_H}{\beta}. \tag{7}$$

In an inhomogeneous magnetic field ω_H, $v_\parallel$ and k are functions of the coordinate z along the magnetic field $\boldsymbol{H}$. The electrons with different $v_\parallel$ interact with a given wave at different point along the magnetic field line.

The quantitative description of cyclotron electron-whistler interactions, including non-linear effects, is based on the self-consistent system of equations for the wave field and for the distribution function of energetic electrons. The field equation for the slowly varying magnetic field can be written as [12]

$$\frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{v}_g \frac{\partial \boldsymbol{B}}{\partial z} = \frac{2\pi\, v_g}{c} \boldsymbol{J}_R, \tag{8}$$

where $\boldsymbol{v}_g$ the group velocity and $\boldsymbol{J}_R$ the current density of resonant electron, obeying equation (7). J_R can be written as

$$J_R = -e\int_0^\infty u_\perp du_\perp \int_{-\infty}^{\infty} dv_\parallel \int_0^{2\pi} d\phi u_\perp F e^{i(\phi-\psi)}, \tag{9}$$

where $v_\perp = u_\perp e^{i\varphi}$ is the perpendicular component of the velocity v (perpendicular to the geomagnetic field lines), φ (z,t) is the contribution of the resonant electrons to the whistler phase, ψ is the contribution to the whistle wave phase due to current of cold plasma $\frac{\partial\psi}{\partial t} = -\omega, \frac{\partial\psi}{\partial z} = k$; ω and k are related by the dispersion relation. The real part of the dispersion relation from eq. (5) is given as

$$\frac{k^2c^2}{\omega_p^2} + \frac{k^2 v_{T_\parallel}^2}{(\omega-\omega_H)^2}\left[A_T + \frac{\omega}{\omega-\omega_H}\right] + \frac{\omega}{\omega-\omega_H} = 0. \tag{10}$$

The above equation can be rewritten as

$$k = \frac{\omega}{c}\mu, \tag{11}$$

where

$$\mu = \left[\frac{c^2\omega_p^2(\omega_H - \omega)}{c^2\omega(\omega-\omega_H)^2 + v_{T_\parallel}^2\omega\omega_p^2\left(A_T - \dfrac{\omega}{\omega-\omega_H}\right)}\right]^{1/2}$$

is refracting index for whistler mode wave propagation. Combining eqs.(7) and (11), the resonance velocity $v_\parallel$ for $n = 1$ (fundamental resonance) is obtain as

$$v_\parallel \equiv v_{R\parallel} = \frac{2c\mu \pm \left[4c^2\mu^2 + 4c^2\left(\dfrac{\omega_H^2}{\omega^2} - 1\right)\left(\mu^2 + \dfrac{\omega_H^2}{\omega^2}\right)\right]^{1/2}}{2\left(\mu^2 + \dfrac{\omega_H^2}{\omega^2}\right)} \tag{12}$$

Eq. (12) yields two values of $v_{R\parallel}$. For the whistler mode propagation, $\omega << \omega_H$ and eq.(7) shows that $v_\parallel$ is negative (implying that the resonantly interacting electrons and whistler waves move in opposite direction as assumed earlier). Hence, we consider –ve sign in eq. (12).

The parallel component of energy of resonant electron $[W_\parallel = m\, v_{R\parallel}^2/2]$ as a function of frequency and L-value at $k_BT_\parallel = 0, 5, 10$ keV is shown in Figure 1. The $W_\parallel$ decreases

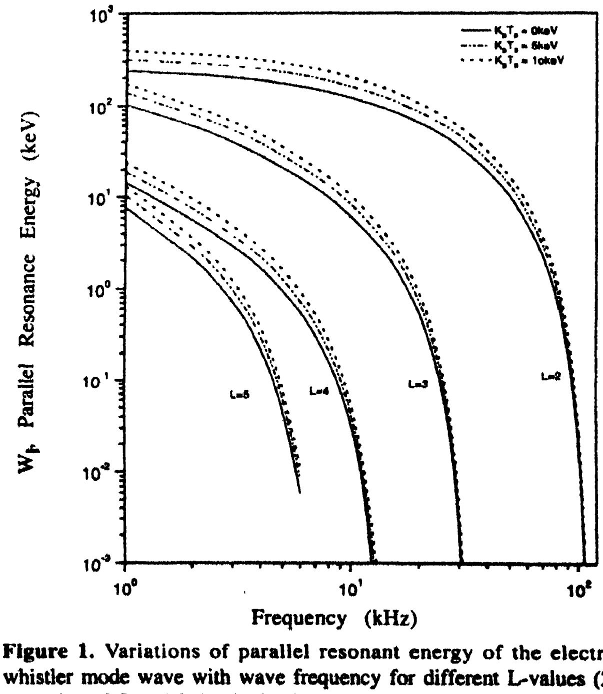

Figure 1. Variations of parallel resonant energy of the electrons of whistler mode wave with wave frequency for different L-values (2, 3, 4, 5) at $A_T = 0.5$ and $k_BT_\parallel = 0, 5, 10$ keV.

with increasing frequency. For lower value of the frequency, contribution due to $k_B T_{\parallel}$ (parallel component of electron thermal velocity) is higher (10 to 20%) in $W_{\parallel}$ but at higher frequency its contribution is only about 2 to 5%. As the values of $k_B T_{\parallel}$ increases, $W_{\parallel}$ also is increases. For a given frequency $W_{\parallel}$ decreases with increasing L-values. For example, the values of $W_{\parallel}$ at L = 2, 3, 4 and 5 (f = 2 kHz and $k_B T_{\parallel}$ = 0 keV) is 219.08 keV, 60.25 keV, 5.788 keV and 2.278 keV respectively. As the L-value varies from 2 to 5, the frequency of the interacting wave varies between 1 kHz to 109 kHz while the parallel resonant electron energy $W_{\parallel}$ may decrease from 3.61×10^2 to 10^{-3} keV. Rice and Hughes [13] have also shown that the resonant electron energy decreases as frequency and L-value increases. Similar behaviour has also been reported by Singh *et al* [6]. Thus, it is clearly seen that in the inner plasmasphere energetic electrons are actually participating in the resonant interaction with the whistler mode wave and, in turn cause emission of the ELF/VLF waves. Singh *et al* [6] reported the variation of resonance electron energy with pitch angle for different L-value and wave frequency, as pitch angle increases resonant electron energy increases but the over all pitch angle dependence is non-linear. We have taken only the normal profile of the equatorial electron density and investigated the variation of $W_{\parallel}$ with frequency at different L-values and various $k_B T_{\parallel}$ (0, 5, 10 keV). It is clearly seen from the Figure 1 is that in order to have an interaction at 2 kHz the gyroresonant energy at L = 4 (just inside the plasmapause) is few keV, whereas deeper within the plasmasphere (at L = 3) it is few tens of keV. Rycroft [14] reported similar result for resonant electrons at L = 3 to 6.

Considering inhomogeneous medium the minimum resonant energy of the electron responsible for emission is given by [15]

$$W_R = \frac{1}{2} mc^2 \left(\frac{\omega_H^3}{\omega \omega_P^2} \right) \left(1 - \frac{\omega}{\omega_H} \right)^2 \left[9 + \left\{ 1 - \frac{L^2}{L_n^2} \right\} \left\{ 1 - \frac{\omega}{\omega_H} \right\} \right], \tag{13}$$

where the parameter L_n is defined through the expression of cold plasma density, [$N(s) = N_0 (1 + s^2/L_n^2)$, where $N(s)$ is cold plasma density at the arc length s from the equator along the field line and N_0 the cold plasma density at the equator]. In the diffusive equilibrium model, L/L_n is very small [15]. The variation of minimum resonant energy (W_R) with wave frequency (for the same L-values as taken in Figure 1) is shown in Figure 2 for L/L_n= 0.05. We observe

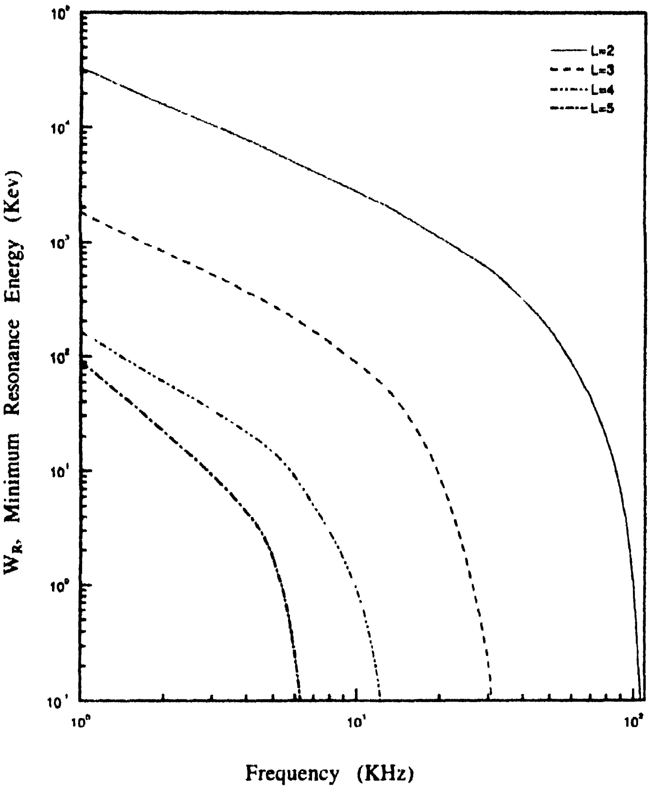

Figure 2. Variations of minimum resonant energy of the electron of whistler mode wave with wave frequency at L = 2, 3, 4, 5 for L/L_n = 0.05.

that W_R is almost 10–20 times greater than the $W_{\parallel}$. We have also calculated the W_R for different values of L/L_n. It is seen that even when L/L_n is changed from 0 to 0.5, there is no appreciable change in minimum resonance energy of the electrons. Molvig *et al* [15] have argued that the most important energies in the emission process are larger than the minimum energy given by eq. (12), in some cases by as much as a factor of 5. Hence, the relevant electron energy may become several tens of keV. Singh [16] has reported the variation of W_R with wave frequency for low latitude ground station of Varanasi (L = 1.07, geoma. Lat. 14°55'). He found that W_R is almost ten times to that of $W_{\parallel}$ value and the shape of the variation with frequency remains the same as that of $W_{\parallel}$.

(ii) Growth rate and amplification of the whistler waves :

The normalized wave growth rate of the whistler mode wave can be written as [9]

$$\gamma = \frac{\dfrac{1}{K_2}\left(\dfrac{\pi}{2}\right)^{1/2}\left(A_T - \dfrac{x}{1-x}\right)(1-x)^3 \exp\left[-\dfrac{(1-x)^2}{2K_2^2}\right] + K_2 \dfrac{eE}{m\omega_H v_{t_\parallel}}\left(A_T + \dfrac{2x}{1-x}\right)}{1 + \dfrac{K_2^2}{(1-x)^2} - \dfrac{2K_2^2}{1-x}\left(A_T - \dfrac{x}{1-x}\right)} \tag{14}$$

where $x = \omega/\omega_H$, $K_2 = kv_{t\parallel}/\omega_H$. The normalized wave growth of the whistler mode wave for various wave frequencies at different L-values (2, 3, 4 and 5) have been computed using the eq. (14). Figure 3 shows the variation of normalized wave growth/damping with the frequency for different dc electric field (E = 0, ± 20, ± 40, ± 60 mV/m) at K_2 = 0.2, A_T = 0.5. In the absence of dc field *i.e* E = 0 mV/m, the wave grow for the lower frequency and exponentially decays at higher frequency up to a critical frequency. After critical frequency amplitude of the wave again increases exponentially. The wave growth/damping depends on the nature of the dc electric field. If the applied field

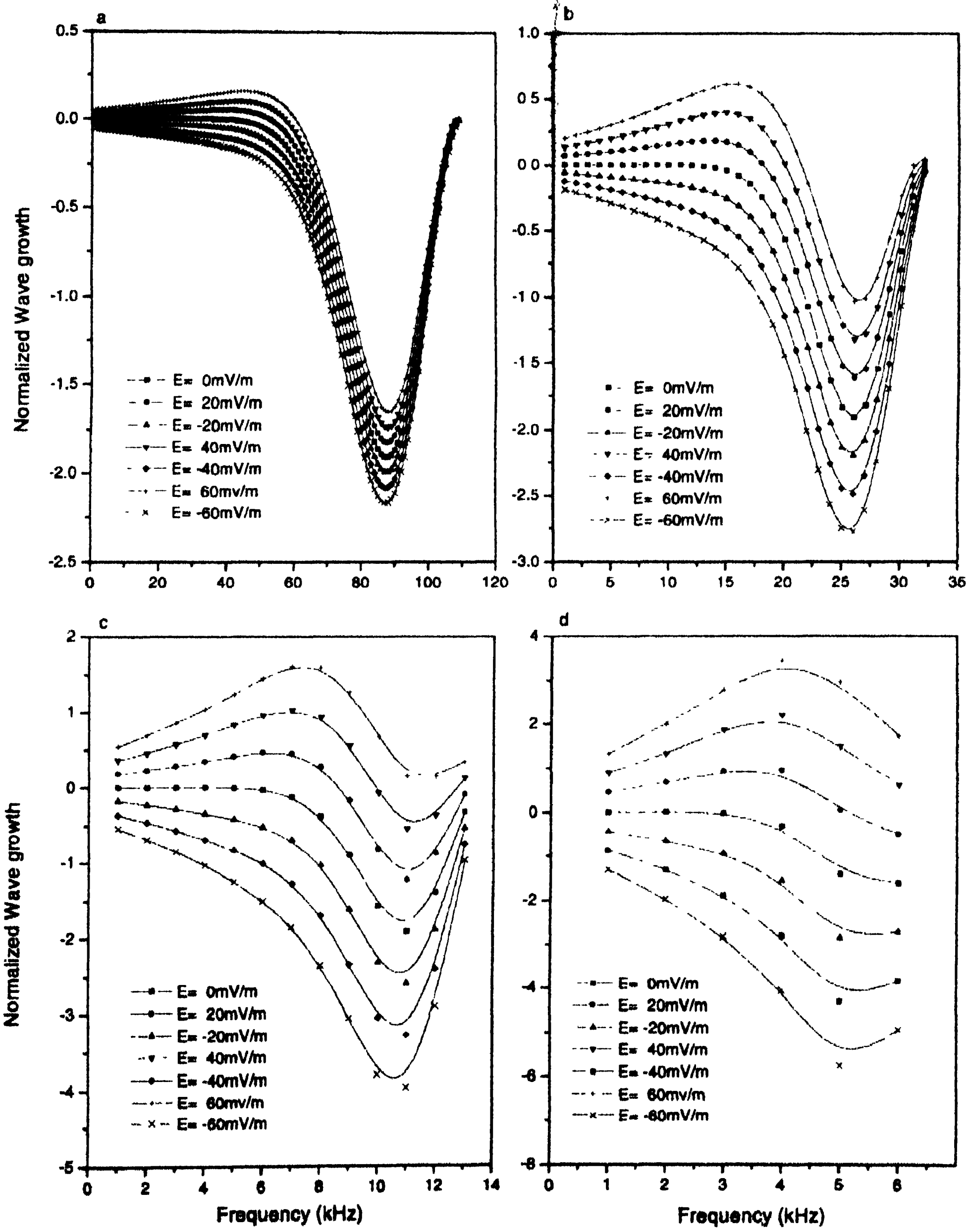

Figure 3. Variations of normalized wave growth of the whistler mode wave with wave frequency, E = 0, ±20, ±40, ±60 mV/m at A_T = 0.5, K_2 = 0.2; (a) L = 2, (b) L = 3, (c) L = 4 and (d) L = 5.

is parallel to magnetic field, waves grow/damp (exponentially) in lower/higher wave frequency. After the critical frequency, wave again starts growing exponentially. If the dc field is antiparallel to the magnetic field, the wave always damps till the critical wave frequency is reached after which it starts growing. All the curves converge to a point near the electron gyrofrequency. Due to the presence of electric field in magnetosphere, the charge particles accelerates during resonance interaction and transfer energy and momentum to the interacting wave leading to wave growth. If the direction of electric field is reversed, charged particles acquire directed velocity in the opposite direction, leading to decay of wave amplitude during wave particle interaction.

The normalized wave growth rates have also been calculated for different K_2 values. The variation normalized wave growth with K_2 is shown in Figure 4 for different values of parallel electric field (E = 0, ± 20, ± 40, ± 60 mV/m) at A_T = 0.5, f = 2 kHz. We find that as the K_2 increases, the magnitude of the normalized wave growth/damping also increases for all the L-value. The shape of the curve for growth rate variation with wave frequency is similar to the observed frequency spectrum in the VLF range [17]. Similar behaviour have been found by Singh *et al* [18] for L = 1.07 and L = 4. From Figure 4 we see that waves are only growing exponentially both for the parallel/antiparallel electric fields deeper inside the plasmasphere (L = 2) after K_2 = 0.45. Although the growth rate for antiparallel electric

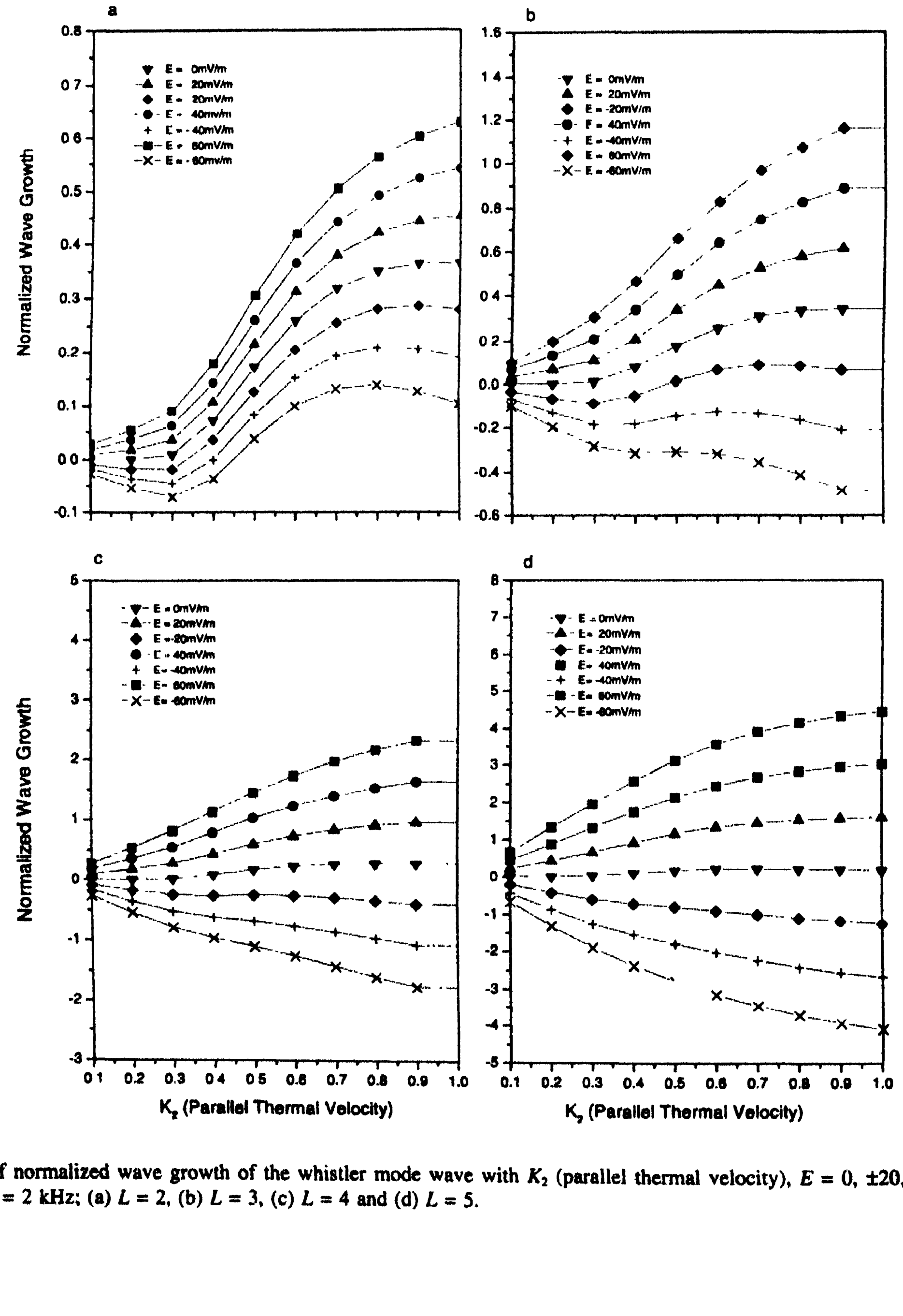

Figure 4. Variations of normalized wave growth of the whistler mode wave with K_2 (parallel thermal velocity), E = 0, ±20, ±40, ±60 mV/m at A_T = 0.5, wave frequency = 2 kHz; (a) L = 2, (b) L = 3, (c) L = 4 and (d) L = 5.

fields are small as compared with the parallel electric fields. As the L-value is gradually increased wave growth for antiparallel electric fields decreases rapidly and only small antiparallel electric fields are able to cause wave growth for increased parallel thermal velocity. For example, deeper within the plasmasphere (L = 3) only $E = -20$ mV/m is able to help in the wave growth after K_2 = 0.5. Inside the plasmapause (L = 4 and 5) in the presence of antiparallel electric fields, waves are damped. The magnitude of the wave growth depends upon the relative magnitude of K_2 and A_T apart from the dc electric field [9].

The amplitude of the interacting wave increases exponentially with time and the amplitude may suddenly build up leading to wave observation on the ground. The wave amplification factor is

$$\alpha = \exp\left(2\int \frac{\gamma_{gt}\, ds}{v_g}\right), \tag{15}$$

where $ds = \frac{R_e}{\cos^2\phi_0}\left(1+3\sin^2\phi\right)^{1/2}\cos\phi\, d\phi$; R_e= 6370 km is the equatorial radius of the Earth, ϕ_0 the magnetic latitude where the field line intersects the Earth surface, ϕ the magnetic latitude, γ_{gt} is the growth rate and $v_g = \left(\frac{2cf^{1/2}(f_H - f)^{3/2}}{f_H f_P}\right)$. We have computed the amplification factor for different L-values. The computed amplification factors are 1.41, 1.98, 25.3, and 2.98 ×10³ for L = 2, 3, 4 and 5 respectively, at f = 2 kHz. For this computation, we have considered that all electrons lying in the flux tube between ±20° from the equator are radiating in phase for L = 5. The length of radiating electron flux tube is considered to be ±15° for the L = 4, ±12.5° for L = 3 and ±10° for L = 2 respectively. Singh *et al* [18] and Singh [16] have computed the amplification factor of the propagating whistler mode ELF/VLF hiss 5 kHz wave frequency is 1.3 for L = 1.07 and 31.5 for L = 4 in the equatorial plane, which is less than the required value to explain to observed spectral power. The wave has to bounce back and forth along the geomagnetic field line many times and it is amplified each time it passes through the equatorial region [16,18]. Helliwell [19] has stressed that ducted signals may echo repeatedly back and forth over one duct for hundred of times. Huang *et al* [20] have claimed that the net growth rates are too small and that the cyclic waves do not provide a satisfactory explanation for hiss generation. Thorne *et al* [21] compute the value of $\alpha \geq 20$ during the disturbed magnetospheric condition and $\alpha \leq 10$ during the quite condition for plasmaspheric ELF hiss. It may be noted that Kennel and Petschek [7], Thorne *et al* [21] and Singh *et al* [18] have not considered the presence of parallel electric field. The presence of parallel electric field enhances the wave growth rate sufficiently even in the absence of thermal anisotropy [9]. This result can be applied to explain the observed average triggering time for the artificially stimulated VLF emissions and naturally occurring VLF emissions. Also, the enhancement of amplitude due to the presence of parallel electric field shows the possibility of observing the events which otherwise would not have been observed. Furthermore, the presence of a parallel electric field causes an increase in the instability range of whistler wave in k-space, and hence the parallel particle velocity needed for resonant interaction decreases, which in turn, increases the available number of resonant interacting particles, leading to higher amplification factor in smaller duration.

4. Conclusion

The relation for parallel resonance energy of the electrons interacting with whistler wave has been derived. The parallel resonance energy of the electron decreases with increase in frequency as well as with L-value but it increases with parallel thermal velocity. We have computed the normalized wave growth and amplification factor at different L-values for various values of dc electric fields (both for parallel as well as antiparallel orientation). Usefulness of the present investigation to the study of natural ELF/VLF emissions and artificially triggered VLF emissions has been pointed out.

Acknowledgments

One of the authors (DKS) is thankful to Head, I & OT Division, Indian Institute of Tropical Meteorology for his encouragement and help throughout the preparation of this manuscript. The authors are also grateful to the anonymous referee for his constructive and valuable suggestion.

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